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An Operational Approach for Testing the Postulate of Measurement in Quantum Theory

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We interpret the (formal) postulates of measurement in quantum theory in terms of measurement procedures that can be done in the laboratory (at least in principle).

1. INTRODUCTION

In 1969 Lamb published an article[(1)] in which he expressed his dissatisfaction with the available discussions of quantum mechanical measurement; he claimed that they are "either too vague" or "too formal and unphysical," and then continued, "in discussion of the measurement of some dynamical variable of a physical system I want to know exactly what apparatus is necessary for the task and how to use it, at least in principle. I am not satisfied with hand waving or with formal logical scheme involving black boxes" (p. 23). Then he pointed out that in order to generalize Bohm's analysis[(2)] (which is not formal and unphysical) for any real dynamical variable $A(x, p)$ we should be able to realize Hamiltonians that are general functions of x and p. This is not a trivial demand, because, as is well known, the potentials that "nature provides" are very restrictive in the variable p.

The realization of this type of Hamiltonian was suggested by Aharonov and Lerner[(3),3] and it is our purpose here to give the operational recipe that follows from their prescription. This will be the recipe for the measure-

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[3] For a brief review suitable for our purpose see Ref. 3, Appendix I.

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ment of any dynamical variable to which there corresponds a Hermitian operator with discrete eigenvalues (and we assume, for simplicity, that any degeneracy has been lifted).

2. THE RECIPE

"Von Neumann's conception of the measurement problem became the framework of almost all subsequent theories of measurement."[(4)]

Bohm[(2)] clarified some of the unpleasant features of this approach[4] by giving a detailed analysis of the measurement of spin by means of a Stern-Gerlach experiment. Both von Neumann's approach and Bohm's analysis are in the background of our presentation, which follows.

The measurement of a dynamical variable $A(x, p)$ will be achieved by correlating the eigenvalues (of the Hermitian operator that corresponds to the above dynamical variable) to the spatial coordinates of a measuring device (obviously, the coordinates of this device are macroscopically distinguishable).

In this approach a localized free particle will stand for the measuring device, namely an incoming particle (let us say in the X direction) will interact with the system for a very short period of time by a coupling term of the form $\delta(t - t_0)\, A(x, p)Z$, where δ is the Dirac delta function and t_0 stands for the measurement time, and Z is the coordinate of the incoming particle (perpendicular to the incoming direction). This is an impulsive interaction[(5)] and therefore the Hamiltonian of the system for a short time is essentially the above coupling term; this term is "responsible" for the "deflection" of the incoming particle according to the values of $A(x, p)$, as can be seen from the following:

$$F_z = \dot{p}_z = -\partial H/\partial z = -\delta(t - t_0)\, A(x, p) \tag{1}$$

in complete analogy to the spin case.[(2)] The difference is that we still have *to specify how to prepare in the laboratory (at least in principle) a force field of the quite general form* $\delta(t - t_0)\, A(x, p)$,[5] while in the spin case (or angular momentum) there is a well-known procedure.[(2)]

Aharonov and Lerner have shown[(3)] that a "respectable" Hamiltonian (quadratic in the momenta) of the form

$$H = (\Omega/2)[(p_x - y/2)^2 + (p_y + x/2)^2] + \bar{A}(x, y) \tag{2}$$

[4] See Ref. 5 for further elaboration and clarification of von Neumann's approach to measurement.

[5] We shall also take into account the fact that, in general, $A(x, p)$ is not a constant of the motion.

will generate, in the limit $\Omega \to \infty$, the dynamics associated with the Hamiltonian $H = A(x, p)$, where the function A is related to the function $\bar{A}$ as follows (see Appendix):

$$A(x, y) = \int \bar{A}(x - x_0, y - y_0) \exp[-(x_0^2 + y_0^2)/\hbar] \, dx_0 \, dy_0 \tag{3}$$

This means that we can "mimic" the force field given by (1) by the above prescription, and the incoming particle will "see" a force field as if produced by the dynamical variable $A(x, p)$.

The following Hamiltonian will allow us to tie together von Neumann's prescription and Bohm's analysis:

$$H = H_0 + \frac{x^2}{2m} + U(y) + \frac{\Pi^2}{2M} + \delta(t - t_0) \bar{A}(x, y) z \tag{4}$$

where $H_0 = \frac{1}{2}\Omega[(p_x - \frac{1}{2}y)^2 + (p_y + \frac{1}{2}x)^2]$ and $\Omega \to \infty$. This means that until the time of measurement t_0,[6] we imitate the time evolution of $A(x, p)$ as dictated by a Hamiltonian of the form $H = p^2/2m + U(x)$; at the time of the measurement the measuring device (which is the free particle, described by $H = \Pi^2/2M$) "sees" the force field $\delta(t - t_0) [H_0 + \bar{A}(x, y)]$, which is equivalent to the force field $\delta(t - t_0) A(x, p)$, and for $t > t_0$ we concentrate on the "deflection" of this particle, in analogy to the measurement of spin or angular momentum.[(2)]

3. CONCLUSION

We have shown that it is possible to "manufacture" a physical system that is equivalent (essentially in every aspect) to a system described by a Hamiltonian $H = p^2/2m + U(x)$ for the purpose of measuring any variable of this system. The measurement is performed on the "replica," but the result is as if the measurement was made on the original.[7] In other words, we have found a way to imitate the "kick" that a measuring device would feel (and register) if it were coupled to the system.

Quoting Lamb once more, we have shown "that the usual textbook assumptions about measurement have meaning" (Ref. 1, p. 28).

[6] The preparation of the state $|\psi(0)\rangle$, which comes even before this stage, can be done, for example, as suggested by Lamb[(1)] or, if we follow our method, by measuring the projection operator $|\psi(0)\rangle\langle\psi(0)|$.

[7] This can be generalized to the three-dimensional case without much difficulty.

APPENDIX

Aharonov and Lerner have recently shown[3] that a gauge-type coupling can be the basis for a renormalization phenomenon in which ordinary configuration space becomes effectively canonically conjugate. Their basic Hamiltonian is of the form

$$H = (\Omega/2)[(p_x - y/2)^2 + (p_y + x/2)^2] + \bar{V}(x, y) \tag{A1}$$

Note that the first term in this Hamiltonian describes essentially the two-dimensional motion of a charged particle interacting with a constant, uniform magnetic field in the z direction. Then, after introducing the definitions

$$x = \bar{x} + \delta x, \qquad y = \bar{y} + \delta y \tag{A2}$$

where

$$\bar{x} = \tfrac{1}{2}x - p_y\,; \qquad \delta x = \tfrac{1}{2}x + p_y\,; \qquad \bar{y} = \tfrac{1}{2}y + p_x\,; \qquad \delta y = \tfrac{1}{2}y - p_x$$

They rewrite their basic Hamiltonian as follows:

$$H = \tfrac{1}{2}\Omega[(\delta x)^2 + (\delta y)^2] + V(\bar{x}, \bar{y}) + \text{“perturbative terms”} \tag{A3}$$

where

$$V(\bar{x}, \bar{y}) = \int \bar{V}(\bar{x} - x_0\,, \bar{y} - y_0) \exp[-(x_0^2 + y_0^2)/\hbar]\, dx_0\, dy_0 \tag{A4}$$

They note that $[\bar{x}, \bar{y}] = i\hbar$, and that $V(\bar{x}, \bar{y})$ commutes with the first term in the Hamiltonian (A3) [this follows from the definitions (A2)].

The “perturbative terms” are terms of the form

$$[e^{i\alpha\delta x} - \langle e^{i\alpha\delta x}\rangle]\, e^{i\alpha\bar{x}} \tag{A5}$$

where the expectation value is taken in the state $N = 0$ [$|N\rangle$ are the eigenstates of the two-dimensional oscillator in (A3)]; the term in square brackets in (A5) obviously has zero expectation value in the $N = 0$ state; it is also a bounded operator, so that, regarded as a perturbation, it gives corrections of order $1/\Omega$, and for $\Omega \to \infty$ these corrections are negligible; therefore the finite excitations of the system will have the spectrum that belongs to the operator $V(\bar{x}, \bar{y})$. This means that if we wish to realize a Hamiltonian of the general form $H(x, p)$, we shall choose the function $\bar{V}(x, y)$ in (A1) such that the function V in the variables $\bar{x}, \bar{y}$ will be the same function as H in the variables x, p. The relation between V and $\bar{V}$ (and therefore between $\bar{V}$ and H) is given in (A4), or also by the following equivalent prescription:

Given the operator $V(x, p)$, we use Weyl's correspondence rule[6]:

$$V(x, p) = \int f(\alpha, \beta) \exp[-(i/\hbar)(\alpha x + \beta p)]\, d\alpha\, d\beta$$
$$f(\alpha, \beta) = \mathrm{Tr}\{V(x, p) \exp[(i/\hbar)(\alpha x + \beta p)]\} \tag{A6}$$

in order to find $f(\alpha, \beta)$; with this information we can write the potential of Eq. (1) as follows:

$$\bar{V}(x, y) = \int f(\alpha, \beta) \exp[(\alpha^2 + \beta^2)/4\hbar] \exp[-(i/\hbar)(\alpha x + \beta y)]\, d\alpha\, d\beta$$

and it is simple to verify that by "renormalizing" $\bar{V}$ [in the sense of Eq. (A4)] we get

$$V(\bar{x}, \bar{y}) = \int f(\alpha, \beta) \exp[-(i/\hbar)(\alpha\bar{x} + \beta\bar{y})]\, d\alpha\, d\beta$$

as desired, because this exactly mimics $V(x, p)$ as given in (A6).

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