Meaning of an individual "Feynman path"

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In this article we give an operational meaning to an individual "Feynman path." In other words, we describe a process of dense measurements, made in temporal sequence, which check whether the particle moves along any given trajectory in space-time. We show that in this process the two assumptions of the space-time formulation of quantum mechanics, are realized: (a) The weight that the particle moves along a trajectory that has been checked by this process is the same for all trajectories, and in fact, we show that the particle follows, with probability 1, the trajectory that is being checked. (b) A phase is systematically accumulated, so that, at the end of this process, the state is multiplied by the familiar factor $\exp[(i/\hbar)\int L\,dt]$. As an immediate extension of the above formalism, we suggest a setup that measures the relative phase between any two trajectories. Finally, our approach points toward the possibility of extending the Feynman formalism in order to cover more general Hamiltonians.

INTRODUCTION

In 1948 Feynman published what is essentially a third formulation of quantum mechanics.[1] As is well known, the main idea in this formulation is to associate a probability amplitude, $\exp[(i/\hbar)\int L\,dt]$, with each possible classical trajectory that connects two space-time points [L is the classical Lagrangian, and the integral is evaluated along the path $X(t)$]. Each possible trajectory is assigned the same weight, and the sum (integral) over the contributions from all possible trajectories has to be carried out in order to get the transition amplitude between these two space-time points.

The concept of a trajectory in quantum mechanics is not a straightforward one because of the uncertainty principle involved.[2] Therefore, attempts have been made to apply the notion of continual observation[3] (which was mentioned by Feynman[1], p. 370) in order to investigate the operational meaning of the trajectories, which are the building blocks of this formalism.

This notion of continuous observations (or measurements) in quantum mechanics has recently attracted some attention,[3-7] because of the interesting features that were revealed. In particular, the following paradoxical property of such measurements was found: Consider the case where repeated observations are carried out in order to find the exact moment at which a transition from some initial state takes place. It turns out that, because of these observations, the transition never occurs. A particular example is the decay of an unstable system[4]: If the system is continuously observed, then it will never decay. Another example is the one of continuously observing a system that is initially confined to a finite space region,[5] and because of these observations it remains confined there. In this paper, we first show that the above paradoxical situation is a special case of a more general property of continuous measurements. Namely, if one checks by continuous observations if a given quantum system evolves from some initial state, to some other final state, along a specific trajectory in Hilbert space, the result is always positive, whether or not the system would have done so on its own accord.

When the above result is applied to the evolution of a state along a trajectory considered by Feynman, we find that the particle follows, with certainty, the trajectory that is being checked. Therefore, it is now meaningful to consider measurements of individual trajectories in space-time and their properties. In particular, the phase associated with the probability amplitude for motion along a given trajectory can be evaluated.

When this calculation is carried out, the phase turns out to be the one assumed by Feynman, (within a constant independent of the trajectory).

The possibility is therefore open to consider setups that measure directly the relative phase of any two individual trajectories (which have common end points); we describe in detail such a setup.

Finally, we point out that our analysis can be applied to more general trajectories in Hilbert space than those corresponding to the classical trajectories. We discuss briefly the relevance of this to the question of extending the Feynman formalism in order to cover arbitrary Hamiltonians.

CONTINUOUS MEASUREMENTS ON A SPIN-HALF SYSTEM

Consider a spin-half particle placed in a constant magnetic field pointing in the z direction with the initial direction of the spin in the $+x$ direction. The time evolution in this case is simply a rotation in the xy plane with the Larmor frequency (defined by $H=\vec{\mu}\cdot\vec{B}\equiv\frac{1}{2}\hbar\omega\hat{\sigma}_z$). If we now want to check when the spin moves out of its initial orientation by performing a dense set of measurements (in time) of $\hat{\sigma}_x$, we find that it does not move out at all (in analogy to the results discussed in the literature[3-7]). This can be seen as follows: In the infinitesimal time δt the free Hamiltonian rotates the direction of the spin, or equivalently changes the state by

$$|\sigma(\delta t)\rangle=\exp\left(-\frac{i}{\hbar}\hat{H}\delta t\right)|\sigma_x=+1\rangle$$
$$=\exp\left(-\frac{i}{2}\omega\hat{\sigma}_z\delta t\right)|\sigma_x=+1\rangle$$
$$=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ e^{-i\omega\delta t}\end{bmatrix},$$

where

$|\sigma(\delta t)\rangle$ is the eigenstate of the operator,

$\hat{\sigma}(\delta t)=\hat{\sigma}_x(0)\cos\omega\delta t+\hat{\sigma}_y(0)\sin\omega\delta t$,

with the eigenvalue plus one.

If we now measure $\hat{\sigma}_x$, the probability that the state collapses to $|\sigma_x=+1\rangle$ is

$$P_r(\sigma_x=+1)=|\langle\sigma_x=+1|\sigma(\delta t)\rangle|^2$$
$$=\cos^2\frac{\omega\delta t}{2}\simeq 1-\frac{(\omega\delta t)^2}{4}\quad(\text{for } \omega\delta t\ll 1).$$

If we repeat the same measurement at intervals of δt the probability that all of them will give the same result is obviously $[1-(\omega\delta t)^2/4]^N$. We now note that if $\delta t=T/N$, where T is the total period of observation, and we approach the limit of very dense measurements ($N\to\infty$), we end up freezing the state in its initial value $|\sigma_x=+1\rangle$, since

$$\lim_{N\to\infty}(1-1/N^2)^N=\lim_{N\to\infty}\exp(-1/N)=1\,.$$

For future reference we refer to this as *case a*.

Case b. Still with the same system as before, we show how it is possible to bypass the feature of freezing while insisting on continuous observations. To achieve this we use the so-called "deterministic observations,"[8] namely, we measure the dense set of operators defined as follows:

$$\hat{\sigma}_n=\hat{\sigma}_x\cos\alpha_n+\hat{\sigma}_y\sin\alpha_n\,,$$

where $\alpha_n=\omega n\delta t$, ω is the Larmor frequency, and $n=1,\ldots,N$. We then obviously find $\sigma_n=1$ for $n=1,\ldots,N$. Thus it seems that we have found a way to monitor the time evolution of a system without freezing it in its initial state.[9]

Case c. In this case we show that the seemingly innocent deterministic observations described above have unsuspected features. Consider again the spin-half particle, initially with $\sigma_x=+1$, but *without a magnetic field*. If we measure the same dense set of operators as in *case b*, we find the following:

The conditional probability to get $\sigma_n=1$ if $\sigma_{n-1}=+1$ is

$$P_r(n)=|\langle\sigma_n=+1|\sigma_{n-1}=+1\rangle|^2$$
$$=\frac{1+\cos\omega\delta t}{2}=\cos^2(\omega\delta t/2),$$

where we have used the eigenvector of $\hat{\sigma}_n$ (that belongs to $\sigma_n=+1$), namely

$$|\sigma_n=+1\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ e^{i\alpha_n}\end{bmatrix}.$$

Thus, for sufficiently large N, the probability of finding $\sigma_n=+1$ in all the measurements is essentially one. This is so even though no magnetic field was present; therefore, the only reasons for the spin rotation are those measurements. This result is quite surprising because of the accepted assumption that if the outcome of a measurement of some dynamical variable is certain (i.e., with probability one), then the state of the system was not disturbed.

The above conclusion can also be derived by the following simple argument: If we analyze *case a* from a rotating-frame-of-reference point of view, so that in the new frame the inertial field exactly cancels the original constant magnetic field,[10] we end up in the same situation as in *case c* (in which the continuous measurements were made at the rate dictated by the Larmor frequency). Therefore, if we accepted that the measurements in *case a* froze the system in the state $|\sigma_x=+1\rangle$, we end up with the result of *case c* as a necessary consequence.

THE PARTICLE FOLLOWS A TRAJECTORY IN PHASE SPACE

We have shown in a simple example that it is possible to define a set of operators that represent a dense sequence of measurements, so that the initial state of the system evolves along the eigenstates of these operators, and the probability to get each of the corresponding eigenvalues is one. This idea can be generalized to more complicated systems, and we now apply it to obtain an operational definition of a "Feynman path." We recall that in Feynman's formulation of quantum mechanics,[1] trajectories in space-time are assigned equal weights, and each trajectory is multiplied by the factor $\exp[(i/h)\int L\,dt]$, L is the Lagrangian of the system, and the integral is along the trajectory. The sum (integral) of the contributions of all trajectories gives the transition amplitude between the two end points.

As in the case of the spin-half system, we can define a dense sequence of measurements which check whether the particle moves along any given trajectory in space-time. This will lead to a positive answer, i.e., the particle moves along any chosen trajectory with probability one.

Consider the set of projection operators

$$\hat{\Pi}_n = |n\rangle\langle n| \,, \quad \text{for } n = 1, \ldots, N,$$

where $|n\rangle$ (in the x representation) is given by

$$|n\rangle = C \exp\{-[X - X(t_n)]^2/(2\Delta X)^2\} \exp\left[\frac{i}{\hbar} P(t_n) X\right] \,.$$

$X(t_n)$, $P(t_n)$ are evaluated along a particular classical trajectory, at a sequence of times $t_n = n\delta t$, the total period of observation is T and $\delta t = T/N$, C is a normalizing factor, and ΔX is the uncertainty. Note that since the classical trajectory is smooth, $\dot{X}(t_n)$ and $\dot{P}(t_n)$ are well defined and so are $\delta X_n \equiv \dot{X}(t_n)\delta t$ and $\delta P_n \equiv \dot{P}(t_n)\delta t$.

We shall now prove that if the initial state is a localized wave packet around $X(t_0) \equiv X_0$, and we measure the set of operators $\hat{\Pi}_n$ (while letting $N \to \infty$), the initial state will evolve along the eigenstates of $\hat{\Pi}_n$ with probability one. We will also show that the change of the phase associated with the evolving state is well defined in terms of the sequence of operations; this change will be evaluated and shown to be equal essentially to the classical action (divided by $\hbar$).

We first note that in the infinitesimal time δt between any two measurements, the state evolves according to the Schrödinger equation and we can write the following general expression[11]:

$$|\psi(t_n + \delta t)\rangle = |n\rangle \exp\left(-\frac{i}{h}\langle n|\hat{H}\,|n\rangle\delta t\right) + b\,|\psi_n^{\perp}\rangle + O(\delta t^2) \,, \tag{1}$$

where $|\psi_n^{\perp}\rangle$ is a state orthogonal to $|n\rangle$ and b is proportional to δt. If we now measure $\hat{\Pi}_{n+1}$ and get a positive answer (which is indeed the case[12]), we end up in the state $|n+1\rangle$, and the probability amplitude for this transition is

$$\langle n+1|\psi_n(t_n + \delta t)\rangle = \langle n+1\,|n\rangle \exp\left[-\frac{i}{\hbar}\langle n|\hat{H}|n\rangle\delta t\right] + O(\delta t) \,. \tag{2}$$

It then follows that

$$|\psi(\text{final})\rangle = |N\rangle \exp\left[-\frac{i}{\hbar}\sum_{n=1}^{N}\langle n|\hat{H}|n\rangle\delta t\right]\prod_{n=1}^{N}\langle n+1|n\rangle + O(\delta t) \,. \tag{3}$$

For a particle described by a Hamiltonian[13] $H = p^2/2m + V(X)$, the expression $\langle n|H|n\rangle$ turns out to be

$$\begin{aligned}\langle n|\hat{H}|n\rangle &= \langle n|p^2/2m + V(\hat{X})|n\rangle \\ &= \left\langle 0\left|\frac{[p - p(t_n)]^2}{2m} + V(\hat{x} - x(t_n))\right|0\right\rangle \\ &= \frac{p^2(t_n)}{2m} + V(X_n) + \theta(\Delta X^2) + \left\langle 0\left|\frac{\hat{p}^2}{2m}\right|0\right\rangle \,.\end{aligned} \tag{4}$$

In the limit in which $\Delta X \to 0$,[14] the above expression will simply be the classical Hamiltonian evaluated along the classical trajectory $X(t)$ plus a constant term independent of the trajectory.

The scalar product of two adjacent eigenstates in Eq. (2), gives the following contribution to the phase:

$$-\frac{i}{\hbar}\Delta p\frac{X(t_n) + X(t_{n+1})}{2} = \frac{i}{\hbar}P(t_n)\dot{X}(t_n)\delta t + \frac{i}{\hbar}[P(t_n)X(t_n) - P(t_{n+1})X(t_{n+1})] \,. \tag{5}$$

Using expressions (3)–(5), and going to the limit $N \to \infty$ $(\Sigma \to \int)$, we find that the accumulated phase associated with the probability amplitude for motion along the trajectory is the sum of the familiar $(i/\hbar)\int L\,dt$, the end terms in Eq. (5), and a term independent of the trajectory $(\langle 0|p^2/2m|0\rangle T)$; note that the terms proportional to $(\delta t)^2$ are negligible because $N \to \infty$ as $1/\delta t$ and therefore $N \times O(\delta t^2) \to 0$.

SETUPS THAT MEASURE THE RELATIVE PHASE OF ANY TWO TRAJECTORIES

No physical meaning can be attached to the phase assigned to an individual trajectory. However, our approach opens up the possibility of considering experimental arrangements that can measure in principle, the relative phase between any two trajectories.

We shall now write the Hamiltonian describing such a setup. This Hamiltonian has to include the degrees of freedom of the measuring device (MD) (following the well-known approach introduced by Von Neumann[15]), in order to have a full description of the process as well as to prove that, indeed, the particle follows the observed trajectory with probability one:

$$\hat{H}=\hat{H}_0+\frac{(1+\hat{\sigma}_x^{(0)})}{2}\sum_n g(t-t_n)[1-\hat{\pi}_n^{(1)}]\hat{\sigma}_x^{(n)}+\frac{(1-\hat{\sigma}_x^{(0)})}{2}\sum_n g(t-t_n)[1-\hat{\pi}_n^{(2)}]\hat{\sigma}_x^{(n)}+\hat{H}_{\rm MD}\,, \quad (6)$$

where $\hat{\pi}_n^{(1)}$ is the projection operator $|n\rangle\langle n|$, with the state $|n\rangle$ parametrized with the coordinates of the first trajectory, and $\hat{\pi}_n^{(2)}$ is similarly defined. The measuring device is described by the $\sigma_x^{(n)}$ ($n=1,2,\ldots,N$), such that initially they are, say, in the spin-up state, and by a proper choice of the function $g(t)$ (as given afterwards), a registration (a positive result in the measurement of the projection operator $|n\rangle\langle n|$) at $t=t_n=n\delta t$ will be inferred from the fact that the nth spin did not flip.

The function $g(t)$ is defined as follows:

$$g(t)=\begin{cases} g_0 & \text{for } 0\leq t\leq\epsilon \\ 0 & \text{otherwise}\end{cases}$$

and we eventually go to the limit $\epsilon\to 0$, $g_0\to\infty$ while keeping $g_0\epsilon=\pi$. This is an impulsive measurement and therefore $\epsilon\ll\delta t$. H_0 is the Hamiltonian describing the particle and $H_{\rm MD}$ is the Hamiltonian of the apparatus not including the interaction term, (we obviously demand $[\sigma_i^{(n)},H_{\rm MD}]=0$ and $[\sigma_i^{(0)},H_{\rm MD}]=0$, $i=x,y,z$).

$\hat{\sigma}_x^{(0)}$ is the operator describing the spin of a particle in the measuring device that, according to its initial value, the process of dense measurements will evolve, i.e., if $\sigma_x^{(0)}=+1$ the process will evolve along trajectory one, if $\sigma_x^{(0)}=-1$, along trajectory two, and if in a superposition of the two state, the process of measurements will also be a superposition of two processes, in the sense described below. We will show that if the initial state of the whole system is given by

$$|\psi(0)\rangle=\frac{c}{\sqrt{2}}\exp[-(x-x_0)^2/(2\Delta x)^2]\left[\exp\left(\frac{i}{\hbar}xp_0\right)|\sigma_x^{(0)}=+1\rangle\exp\left(-\frac{i}{\hbar}x_0p_0\right)+\exp\left(\frac{i}{\hbar}xp_0'\right)|\sigma_x^{(0)}=-1\rangle\exp\left(-\frac{i}{\hbar}x_0p_0'\right)\right]\prod_{n=1}^{N}|\sigma_x^{(n)}=+1\rangle$$
$$\simeq c\exp[-(x-x_0)^2/(2\Delta x)^2]|\sigma_z^{(0)}=+1\rangle\prod_{n=1}^{N}|\sigma_x^{(n)}=+1\rangle \quad (7a)$$

[the last (approximate) equality is due to $\Delta x\to 0$; remember that we always go to this limit[14]], then the final state will be

$$|\psi(\text{final})\rangle\simeq\frac{c}{\sqrt{2}}\exp\{-[x-x(t_n)]^2/(2\Delta x)^2\}\left[|\sigma_x^{(0)}=+1\rangle+\exp\left(\frac{i}{\hbar}\alpha\right)|\sigma_x^{(0)}=-1\rangle\right]\prod_{n=1}^{N}|\sigma_x^{(n)}=+1\rangle, \quad (7b)$$

where $\alpha\equiv\int_1 L\,dt-\int_2 L\,dt$; this means that the direction of the zeroth spin is sensitive to α.

Let us outline the steps that lead to the above result: First, assume that the initial state (of the whole system) is the one corresponding to the measurement of the first trajectory; then we find that the state evolves as follows:

(a) In the infinitesimal time δt before the first measurement we end up with [see Eq. (1)]

$$|\psi(t)\rangle=\left[|0\rangle\exp\left(-\frac{i}{\hbar}\langle 0|H|0\rangle\delta t-\frac{i}{\hbar}x_0P_0\right)+b|\psi_0^{\perp}\rangle+O(\delta t^2)\right]|{\rm MD}^{(0)}\rangle\,,$$

where $|{\rm MD}^{(0)}\rangle$ means no spins have been flipped in the measuring device, etc.

(b) At $t=\delta t$ we have an impulsive measurement (i.e., only the interaction with the MD is relevant[15]). This changes $|\psi(t)\rangle$ into

$$|\psi(t=\delta t)\rangle=\left[\langle 1|0\rangle\exp\left(-\frac{i}{\hbar}\langle 0|H|0\rangle\delta t-\frac{i}{\hbar}x_0P_0\right)+O(\delta t^2)\right]|1\rangle|{\rm MD}^{(0)}\rangle$$
$$+\left[\langle\psi_1^{\perp}|0\rangle\exp\left(-\frac{i}{\hbar}\langle 0|H|0\rangle\delta t-x_0P_0\right)+O(\delta t)\right]|\psi_1^{\perp}\rangle|{\rm MD}^{(1)}\rangle\,,$$

where $|\psi_1^{\perp}\rangle$ is a state orthogonal to $|1\rangle$.

This procedure can be repeated, i.e., the free Hamiltonian acts on the state (in between measurements), and then the interaction term (which is impulsive) takes over, etc., N times.

Letting $N\to\infty$ and $\delta t\to 0$ while keeping $N\delta t=T$ we realize that the *probability* that one of the spins had been flipped (in the measuring device) is a sum of N terms each proportional to $(\delta t)^2$ (and higher orders), so that $N(\delta t)^2 \to_{N\to\infty} 0$. This is so because of the fact that terms in the amplitude that are proportional to δt are multiplied by orthogonal states of the measuring device. Then, using Eqs. (3)–(5) we can write the final state as follows:

$$|\psi(\text{final})\rangle = |N\rangle |\sigma_x^{(0)}=+1\rangle \exp\left(\frac{i}{\hbar}\int_1 L\,dt\right)\exp\left(-\frac{i}{\hbar}x_N P_N\right)\exp\left(-\frac{i}{\hbar}\langle 0|\hat{P}^2/2m|0\rangle T\right)\prod_{n=1}^{N}|\sigma_x^{(n)}=+1\rangle\,.$$

If we superpose two such states coming from different trajectories and recall that $\Delta X\to 0$ (which will make the end term disappear), we end up with the final state given by Eq. (7b), which means we also observe the relative phase between any two trajectories.

CONCLUSIONS

In this paper we have shown that it is possible to define trajectories in Hilbert space connecting any two states. If one measures successively a dense set of projection operators corresponding to the states that define a trajectory,[16] one moves with probability one from the initial state to the final state through the whole set.

We pointed out that the problem of freezing a state as a result of continuous observations is a particular case of this more general notion of trajectory. We have applied the above notion to give an observable meaning to an individual Feynman path, and we have shown that a phase is systematically accumulated during this process of observation.

An experimental arrangement was suggested in order to measure the relative phase between any two trajectories; this phase was shown to agree with the one postulated by Feynman. The concepts of trajectories and their associated phases can serve as the starting point for an extension of the Feynman formalism to include more general Hamiltonians (see the Appendix). These ideas will be further discussed in a future publication.

We thank Professor E. Lerner for helpful comments.

APPENDIX

Consider, for example, the following trajectory that does not have a classical counterpart. We use the eigenoperators of the two-slit experiment[8] which are defined as follows:

$$\bar{\sigma}_3 = \frac{\sin(\pi\hat{x}/l)}{|\sin(\pi\hat{x}/l)|}\,,$$

$$\bar{\sigma}_1 = \cos(\hat{p}l/h) - \sin(\hat{p}l/h)\bar{\sigma}_3\,,$$

$$\bar{\sigma}_2 = \sin(\hat{p}l/h) + i\cos(\hat{p}l/h)\bar{\sigma}_3\,,$$

where $\hat{x}$ is the position operator along the line connecting any two given points in space, and the origin ($x=0$) is in the middle of the two.

Assuming that the initial state $|0\rangle$ and final state $|N\rangle$ of the previous example are localized wave packets with negligible overlap, we can define a trajectory connecting them, as follows:

Let us call $|0\rangle$ and $|N\rangle$ the "spin-up" and "spin-down" states of $\bar{\sigma}_3$, respectively. Then, if we perform the sequence of measurements $\bar{\sigma}_{\alpha_n}=\bar{\sigma}_3\cos\alpha_n + \sigma_2\sin\alpha_n$, $n=0,1,\ldots,N$, where $\alpha_n=n\pi/N$ and let $N\to\infty$, we find that the initial state $|0\rangle$ will evolve along the trajectory defined by the eigenoperators of $\bar{\sigma}_{\alpha_n}$, finally reaching the state $|N\rangle$.[17] The proof goes in complete analogy to the one we gave for the spin-half system. Note that in this last example the Hamiltonian includes a nonlocal term.[18]

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[8]Y. Aharonov *et al.*, Int. J. Theor. Phys. 3, 443 (1970).
[9]Note that in the case of the decaying system, the deterministic measurements have to involve also the field degrees of freedom.
[10]This is equivalent to the unitary transformation $U=\exp[(i/\hbar)(\omega t\hat{\sigma}_z/2)]$ where $H=\vec{\mu}\cdot\vec{B}\equiv\hbar\omega\hat{\sigma}_z/2$.
[11]This is evident from the following relations:

$\psi(t)=\psi(0)+\dot{\psi}(0)\delta t + O(\delta t^2)$,

$\dot{\psi}(0)=(1/i\hbar)\hat{H}\psi(0)$ (Schrödinger equation at $t=0$); by inserting (b) into (a) we get

$$\psi(t) = \psi(0) + (1/i\hbar)\hat{H}\psi(0)\delta t + O(\delta t^2),$$

$$\langle\psi(0)|\psi(t)\rangle = 1 + (1/i\hbar)\langle 0|\hat{H}|0\rangle\,\delta t + O(\delta t^2)$$
$$= \exp\,[-(i/\hbar)\langle 0|\hat{H}|0\rangle\,\delta t] + O(\delta t^2).$$

Now we write an alternative identity for $|\psi(t)\rangle$, namely,

$$\psi(t) = a\psi(0) + b\,|\,\psi_0^{\perp}\rangle,$$

where $a \equiv \exp[(i/\hbar)\langle 0|H|0\rangle\,\delta t] + O(\delta t^2)$ and $|\psi_0^{\perp}\rangle$ is an orthogonal state to $\psi(0)$. Finally we note that $b \sim \delta t$ since

$$1 \equiv \langle\psi(t)|\psi(t)\rangle = |a|^2\langle\psi_0|\psi_0\rangle + \langle\psi_0^{\perp}|\psi_0^{\perp}\rangle|\,b\,|^2$$

and because $\langle\psi(0)|\psi(0)\rangle \equiv 1$ and $\langle\psi_0^{\perp}|\psi_0^{\perp}\rangle = 1$, $|\,b\,|^2 = 1 - |\,a\,|^2 = O(\delta t^2)$.

[12]See a detailed analysis of the measurement in the last section of the paper, but note that a straightforward proof can be given in analogy to the spin case, where the only differences are the following: (a) Instead of the magnetic field we now have a more general Hamiltonian, but still the probability that the state escapes from the original one is of the order $(\delta t)^2$ (more exactly, $|\langle n\,|\,t\rangle|^2 = 1 - [(\delta t)^2/\hbar^2](\Delta E)^2 + O(\delta t^3)$, where ΔE is the uncertainty of the energy in the state $|\,n\rangle$ and $|\,t\rangle$ is the state between t_n and t_{n+1}). (b) Instead of measuring the same projection operator at intervals δt, we now measure projection operators of states that are "slightly" shifted, namely, $|\langle n+1|n\rangle|^2 \sim (\delta t)^2$. It is easy to show now that $|\langle n+1\,|\,t\rangle|^2 \sim (\delta t)^2$, and from this point the argument follows the one given in the spin case.

[13]It can be generalized straightforwardly to the three dimensional case including a vector potential.

[14]Recall that we demanded $\prod_{n=1}^{N}|\langle n+1|n\rangle^2 \to 1$ as $N\to\infty$, which is satisfied if $[(\delta x)_{\max}/\Delta x]^2 N \to 0$ as $N\to\infty$ (because $|\langle n+1|n\rangle|^2 \sim \exp[-(\delta x/\Delta x)^2]$ and $(\delta x)_{\max}$ is the maximum distance between the centers of two adjacent Gaussians along the trajectory), then using the fact that $N \equiv T/\delta t$ and $(\delta x)_{\max} = \dot{x}_{\max}\,\delta t$ we get

$$\frac{T(\dot{X})_{\max}^2\,\delta t}{(\Delta x)^2}\xrightarrow[\delta t\to 0]{} 0.$$

This means that we first have to go to the limit $\delta t \to 0$ and then to choose Δx arbitrarily small.

[15]See, for example, Y. Aharonov and J. L. Safko, Ann. Phys. (N. Y.) 91, 279 (1975).

[16]In the sense that if a total of N measurements are made (in some given period of time T) then the probability of success in each measurement has to be at least of the order $(1 - 1/N^2)$, and $N \gg 1$.

[17]The particle makes quantum jumps from its initial position to the final position without going through intermediate ones.

[18]The realization of these types of Hamiltonians is considered in a recent paper by Y. Aharonov and E. Lerner, Phys. Rev. D 20, 1877 (1979).